3. Neimark, Iu. I., The method of point mappings in the theory of nonlinear oscillations. Izv. VUZ, Radiofizika Vol. 1, №2, 1958.
4. Komraz, L. A., Bifurcations of the fixed points of a point transformation under which a root of the characteristic polynomial passes through the value $\lambda = -1$. PMM Vol. 32, №3, 1968.

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DETERMINING THE COEFFICIENTS OF THE LEGENDRE POLYNOMIAL EXPANSION OF THE EARTH'S GRAVITATIONAL POTENTIAL

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The solution of the Stokes problem [1−4] is used to find exact expressions for the coefficients of the Legendre polynomial expansion of the potential of the Earth's regularized gravitational field with the Clairaut ellipsoid taken as the equipotential surface.

1. The solution of the Stokes problem with the Clairaut ellipsoid taken as the equipotential surface of the Earth's gravitational field [4] yields the following expression for the potential V of this field [1−3] in the Earth-centered orthogonal coordinate system $Oxyz$ (the origin O of this system coincides with the Earth's center; its z-axis is directed along the Earth's axis of rotation):

$$V(x, y, z) = -AP(x^2 + y^2) - BQz^2 + CR \tag{1.1}$$

Here

$$P = \operatorname{arc\,tg} \varepsilon' - \frac{\varepsilon'}{1 + \varepsilon'^2}, \qquad Q = \varepsilon' - \operatorname{arc\,tg} \varepsilon', \qquad R = \operatorname{arc\,tg} \varepsilon' \tag{1.2}$$

where ε' is the second eccentricity of the ellipsoid which is confocal with the Clairaut ellipsoid and passes through the point at which the potential is being determined; A, B and C are constants.

The quantity ε' is given by the equation

$$\varepsilon' = [(a^2 - b^2) / (b^2 + \nu)]^{1/2} \tag{1.3}$$

where a and b are the major and minor semiaxes of the Clairaut ellipsoid and ν is the positive root of the equation

$$\frac{x^2 + y^2}{a^2 + \nu} + \frac{z^2}{b^2 + \nu} = 1 \tag{1.4}$$

The constants A, B and C appearing in formula (1.1) can be determined from the relations

$$A = \frac{u^2(1 + \varepsilon^2)}{2[(3 + \varepsilon^2)\operatorname{arc\,tg} \varepsilon - 3\varepsilon]}, \qquad B = 2A$$
$$C = \frac{a^2}{\varepsilon}\left\{\frac{g_e + u^2 a}{a} + \frac{u^2(1 + \varepsilon^2)(\varepsilon - \operatorname{arc\,tg} \varepsilon)}{(3 + \varepsilon^2)\operatorname{arc\,tg} \varepsilon - 3\varepsilon}\right\} \tag{1.5}$$

Here $\varepsilon = (a^2 - b^2)^{1/2} / b$ is the second eccentricity of the Clairaut ellipsoid, u is the Earth's angular velocity, and g_e is the gravitational acceleration at the equator.

Relations (1.1)−(1.5) yield an implicit expression for the potential $V(x, y, z)$. This expression is inconvenient for practical computations, which is why the gravitational field potential is usually expressed [2–5] as an expansion in Legendre polynomials,

$$V(r, \varphi)=\sum_{n=0}^{\infty} A_n\left(\frac{a}{r}\right)^{n+1} P_n(\sin \varphi) \tag{1.6}$$

where the distance r from the Earth's center and the latitude φ are the geocentric coordinates of the point at which the potential is being determined, P_n $(\sin\varphi)$ are nth order Legendre polynomials, and A_n are the constant coefficients of the expansion.

Exact solution (1.1)–(1.5) of the Stokes problem is not usually used in determining the coefficients A_n of expansion (1.6). The standard procedure [1, 4] is to determine them directly on the basis of the fact that the surface of the Clairaut ellipsoid is the equipotential surface of the gravitational field. This yields approximate expressions for the first coefficients A_n in the form of series in the small parameters

$$\varepsilon^2=\frac{a^2-b^2}{b^2}\left(\text{ or } e^2=\frac{a^2-b^2}{a^2}\right) \text{ and } q=\frac{u^2 a}{g_e} \tag{1.7}$$

The first (and probably the second) terms of the expansions are relatively easy to obtain [3, 4]. This approach also affords the opportunity of constructing recursion relations for the successive determination of the expansions of all coefficients A_n with an arbitrary degree of accuracy. However, the method of obtaining these relations and the relations themselves are exceedingly cumbersome, since the problem reduces ultimately to the solution of an infinite (triangular) system of linear algebraic equations.

The first coefficients A_n of the expansion of potential (1.6) are determined in [6] on the basis of the exact solution of the Stokes problem, but also in the form of expansions in powers of small parameters (1.7).

2. Converting to the spherical (geocentric) coordinates r, φ, λ in (1.1) by setting $x = r\cos\varphi\cos\lambda$, $y = r\cos\varphi\sin\lambda$, $z = r\sin\varphi$, we obtain

$$V=-A\left(\frac{r}{a}\right)^2(P\cos^2\varphi+2Q\sin^2\varphi)+CR \tag{2.1}$$

Introducing the notation

$$t=\frac{a}{r}, \qquad e=\frac{\varepsilon}{\sqrt{1+\varepsilon^2}} \tag{2.2}$$

(where e is the first eccentricity of the Clairaut ellipsoid), we obtain from (1.3), (1.4) the following equation for determining ε':

$$\varepsilon'^4\sin^2\varphi-\varepsilon'^2(e^2t^2-1)-e^2t^2=0 \tag{2.3}$$

whence we have

$$\varepsilon'=\frac{1}{\sqrt{2}\sin\varphi}\left[e^2t^2-1+\sqrt{(e^2t^2-1)^2+4e^2t^2\sin^2\varphi}\right]^{1/2} \tag{2.4}$$

From the last equation of (1.2) and Eq. (2.4) we find that

$$\frac{\partial R}{\partial t}=\frac{e}{\sqrt{2}}\left[\frac{\sqrt{(e^2t^2-1)^2+4e^2t^2\sin^2\varphi}-e^2t^2+1}{(e^2t^2-1)^2+4e^2t^2\sin^2\varphi}\right]^{1/2} \tag{2.5}$$

Converting to complex numbers, we can rewrite expression (2.5) for $\partial R / \partial t$ as

$$\frac{\partial R}{\partial t}=\frac{e}{2}\left[\frac{1}{\sqrt{1-2\sin\varphi\,(iet)+(iet)^2}}+\frac{1}{\sqrt{1-2\sin\varphi\,(-iet)+(-iet)^2}}\right], \qquad i=\sqrt{-1} \tag{2.6}$$

Since the function $(1-2x\tau+\tau^2)^{-1/2}$ is the generating function of the Legendre polynomials P_n (x), i.e. since

$$\frac{1}{\sqrt{1-2x\tau+\tau^2}}=\sum_{n=0}^{\infty}\tau^n P_n(x) \tag{2.7}$$

it follows from (2.6) that

$$\frac{\partial R}{\partial t}=\frac{e}{2}\sum_{n=0}^{\infty}P_n(\sin\varphi)\,[(iet)^n+(-1)^n(iet)^n]=e\sum_{k=0}^{\infty}(-1)^k(et)^{2k}P_{2k}(\sin\varphi) \quad (2.8)$$

This implies that $(t=a\,/\,r)$

$$R=\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\left(e\frac{a}{r}\right)^{2k+1}P_{2k}(\sin\varphi)+C_1(\varphi) \quad (2.9)$$

where $C_1(\varphi)$ is some function of the lattitude φ.

Next, Eqs. (1.2) and (2.4) with allowance for (2.2) yield

$$\frac{\partial}{\partial t}(P\cos^2\varphi+2Q\sin^2\varphi)=2e^2t^2\frac{\partial R}{\partial t} \quad (2.10)$$

Converting back from t to $a\,/\,r$, we infer from (2.10) and (2.8) that

$$\left(\frac{r}{a}\right)^2(P\cos^2\varphi+2Q\sin^2\varphi)=2e^2\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+3}\left(e\frac{a}{r}\right)^{2k+1}P_{2k}(\sin\varphi)+\left(\frac{r}{a}\right)^2C_2(\varphi) \quad (2.11)$$

where $C_2(\varphi)$ is also some function of the lattitude φ.

3. Substituting expressions (2.9) and (2.11) into Eqs. (2.1), we obtain the following expression for the potential V:

$$V=\sum_{k=0}^{\infty}\left[-\frac{2Ae^2}{2k+3}+\frac{C}{2k+1}\right](-1)^k\left(e\frac{a}{r}\right)^{2k+1}P_{2k}(\sin\varphi)+CC_1(\varphi)-A\left(\frac{r}{a}\right)^2C_2(\varphi) \quad (3.1)$$

Since the potential V satisfies the condition

$$\lim rV=\text{const},\quad r\to\infty \quad (3.2)$$

it follows that $C_1(\varphi)\equiv 0$, $C_2(\varphi)\equiv 0$ in (3.1). This in turn yields the following expression for the Earth's gravitational potential:

$$V(r,\varphi)=\sum_{k=0}^{\infty}\left[-\frac{2Ae^2}{2k+3}+\frac{C}{2k+1}\right](-1)^k\left(e\frac{a}{r}\right)^{2k+1}P_{2k}(\sin\varphi) \quad (3.3)$$

Relations (1.5) and the last equation of (1.7) give us the following values of the constants A and C:

$$A=\frac{g_e qa(1+\varepsilon^2)}{2[(3+\varepsilon^2)\operatorname{arc\,tg}\varepsilon-3\varepsilon]}$$

$$C=\frac{g_e a}{\varepsilon}\left\{1+q\left[1+\frac{(1+\varepsilon^2)(\varepsilon-\operatorname{arc\,tg}\varepsilon)}{(3+\varepsilon^2)\operatorname{arc\,tg}\varepsilon-3\varepsilon}\right]\right\} \quad (3.4)$$

Thus,

$$V=\sum_{k=0}^{\infty}A_{2k}\left(\frac{a}{r}\right)^{2k+1}P_{2k}(\sin\varphi) \quad (3.5)$$

where

$$A_{2k}=(-1)^k e^{2k+1}\left(-\frac{2Ae^2}{2k+3}+\frac{C}{2k+1}\right) \quad (3.6)$$

and where the constants A and C are given by Eqs. (3.4).

Substituting the values of A and C into formula (3.4) and taking account of the second equation of (2.2), we finally obtain

$$A_{2k}=\frac{g_e a\varepsilon^{2k}}{(1+\varepsilon^2)^{1/2(2k+1)}}\frac{(-1)^k}{2k+1}\left\{1+2q\frac{(2k+1)(\operatorname{arc\,tg}\varepsilon-\varepsilon)+\varepsilon^3}{(2k+3)[(3+\varepsilon^2)\operatorname{arc\,tg}\varepsilon-3\varepsilon]}\right\} \quad (3.7)$$

As an alternative to (3. 5) we can also write the expression for the potential in the form

$$V = J_0\left[\frac{a}{r} + \sum_{k=1}^{\infty} J_{2k}\left(\frac{a}{r}\right)^{2k+1} P_{2k}(\sin\varphi)\right] \tag{3.8}$$

Then

$$J_0 = \frac{g_e a}{(1+\varepsilon^2)^{1/2}}\left\{1 + \frac{2q}{3}\,\frac{3(\operatorname{arc\,tg}\varepsilon - \varepsilon) + \varepsilon^3}{(3+\varepsilon^2)\operatorname{arc\,tg}\varepsilon - 3\varepsilon}\right\}$$

$$J_{2k} = \frac{(-1)^k \varepsilon^{2k}}{(2k+1)(1+\varepsilon^2)^k} \times \tag{3.9}$$

$$\times\left\{1 - \frac{4k\varepsilon^3 q}{(2k+3)\left[3(3+\varepsilon^2+2q)\operatorname{arctg}\varepsilon - 9\varepsilon - 6\varepsilon q + 2\varepsilon^3 q\right]}\right\}$$

Clearly, expansion of formulas (3. 9) in powers of ε^2 and q yields the following approximate expressions for the coefficients J_{2k}:

$$\begin{gathered} J_0 = g_e\,(1 - {}^1/_2\varepsilon^2 + {}^3/_2 q + {}^3/_8\varepsilon^4 - {}^{15}/_{18}\varepsilon^6 + {}^{943}/_{2352}\varepsilon^4 q\ldots) \\ J_2 = -{}^1/_3\varepsilon^2 + {}^1/_3 q + {}^1/_3\varepsilon^4 - {}^1/_{21}\varepsilon^3 q - {}^1/_2 q^2 - {}^1/_3\varepsilon^6 + {}^3/_4 q^3 + {}^8/_{147}\varepsilon^4 q\ldots \\ J_4 = \varepsilon^2({}^1/_5\varepsilon^2 - {}^2/_7 q - {}^2/_5\varepsilon^4 + {}^3/_7 q^2 + {}^{16}/_{49}\varepsilon^2 q\ldots) \\ J_6 = \varepsilon^4\,({}^5/_{21} q - {}^1/_7\varepsilon^2\ldots) \end{gathered} \tag{3.10}$$

The first terms of the resulting expansions of J_0, J_2, J_4 coincide with those appearing in [4, 6].

Setting [1, 4, 7, 8] $u = 7.29212 \cdot 10^{-5}$ sec^{-1} and $g_e = 978.049$ cm/sec^2, we obtain the following numerical values of these coefficients for the parameters of the Krasovskii ellipsoid ($a = 6738245$ m, $e^2 = 0.006693422$) :

$$J_0/a = 979.846 \text{ м},\ J_2 = -1082.24 \cdot 10^{-6},\ J_4 = 2.4 \cdot 10^{-6},\ J_6 = -6.3 \cdot 10^{-9};$$

The corresponding values for the parameters of the Clark ellipsoid ($a = 6378206$ m, $e^2 = 0{,}00676817$) are

$$J_0/a = 979.809 \text{ м},\ J_2 = -1107.19 \cdot 10^{-6},\ J_4 = 2.5 \cdot 10^{-6},\ J_6 = -6.3 \cdot 10^{-9}.$$

BIBLIOGRAPHY

1. Mikhailov, A. A., A Course in Gravimetry and the Theory of the Earth's Figure. Editorial Board of the Main Administration of Geodesy and Cartography of the Council of People's Commissars SSSR. Moscow, Gostekhizdat, 1939.
2. Duboshin, N. G., The Theory of Gravitation. Moscow, Fizmatgiz. 1961.
3. Idel'son, N. I., The Theory of the Potential and Its Applications to Geophysical Problems. Leningrad-Moscow, Gostekhizdat, 1932.
4. Grushinskii, N. P., The Theory of the Earth's Figure. Moscow, Fizmatgiz, 1963.
5. Demin, V. G., The Motion of an Artificial Earth Satellite in a Noncentral Gravitational Field. Moscow, "Nauka", 1968.
6. Andreev, V. D., On the solution of the Stokes problem for an equipotential surface specified in the form of a spheroid. PMM Vol. 30, №2, 1966.
7. Graur, A. V., Mathematical Cartography. Izd. Leningr. gos. Univ., 1956.
8. Madelung, E., The Mathematical Apparatus of Physics. Moscow, Fizmatgiz, 1960.

Translated by A. Y.